

WEIGHTED CONVOLUTION INEQUALITIES FOR RADIAL FUNCTIONS

PABLO L. DE NÁPOLI AND IRENE DRELICHMAN

ABSTRACT. We obtain convolution inequalities in Lebesgue and Lorentz with power weights when the functions involved are assumed to be radially symmetric. As a corollary, we also obtain weighted inequalities for Riesz potentials of radial functions in weighted Lorentz spaces.

1. INTRODUCTION

The aim of this paper is to study boundedness properties of the convolution operator

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y) dy$$

in Lebesgue and Lorentz spaces with power weights, when restricted to radially symmetric functions.

To state our results, first we need to introduce some notations. Given a measurable function f in \mathbb{R}^n , we denote its distribution function with respect to the weight $w(x) = |x|^{\alpha p}$ by

$$\mu_f(s) = \int_{\{x: |f(x)| > s\}} |x|^{\alpha p} dx, \quad s > 0.$$

Then, the weighted Lorentz space $L(p, q; \alpha)$ is the space of all measurable functions in \mathbb{R}^n such that $\|f\|_{p, q; \alpha}$ is finite, with

$$\begin{aligned} \|f\|_{p, q; \alpha} &= \left(q \int_0^\infty s^{q-1} \mu_f(s)^{\frac{q}{p}} ds \right)^{\frac{1}{q}}, \quad 1 < p < \infty, 1 \leq q < \infty, \\ \|f\|_{p, \infty; \alpha} &= \sup_{s>0} s \mu_f(s)^{\frac{1}{p}}, \quad 1 \leq p < \infty. \end{aligned}$$

When $p = q$, we recover the weighted Lebesgue space $L(p, p; \alpha) = L(p; \alpha)$ with

$$\begin{aligned} \|f\|_{p, \alpha} &= \left(\int_{\mathbb{R}^n} |f(x)|^p |x|^{\alpha p} dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty \\ \|f\|_{\infty; \alpha} &= \operatorname{ess\,sup}_{x \in \mathbb{R}^n} |f(x)w(x)|. \end{aligned}$$

When $\alpha = 0$ we simply write L^p . Finally, by $L_{rad}(p, q; \alpha)$, $L_{rad}(p, \alpha)$ or L_{rad}^p we denote the subspaces of radial functions of the corresponding spaces.

2010 *Mathematics Subject Classification.* Primary 44A35, 42A85, 26D15. Secondary 47G10, 26A99.

Key words and phrases. Convolution, Young's inequality, weighted inequalities, radial functions, Riesz potentials, fractional integrals.

Supported by ANPCyT under grant PICT 1675/2010, by CONICET under grant PIP 1420090100230 and by Universidad de Buenos Aires under grant 20020090100067. The authors are members of CONICET, Argentina.

Following [6], given functional spaces X, Y, Z , we shall write $X * Y \subset Z$ to indicate that for functions $f \in X, g \in Y$, then $f * g \in Z$ and there exists a positive constant C such that

$$\|f * g\|_Z \leq C \|f\|_X \|g\|_Y.$$

Then, the classical Young's inequality reads

Theorem. (*Young's inequality*)

$$L_p * L_q \subset L_r$$

for $1 \leq p, q, r \leq \infty$, provided that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$.

In Lorentz spaces the result is due to O'Neil [9]:

Theorem.

$$L(p_0, q_0) * L(p_1, q_1) \subset L(p, q)$$

for $1 < p_0, p_1, p < \infty$, provided that $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1} - 1$ and $0 \leq \frac{1}{q} \leq \frac{1}{q_0} + \frac{1}{q_1} \leq 1$.

In the case of power weights, the above theorems were generalized by R. Kerman [6]. Partial results for the L^p case can also be found in [2].

Theorem 1.1. [6, Theorem 3.1]

$$L(p; \alpha) * L(q; \beta) \subset L(r; -\gamma)$$

provided

- (1) $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} + \frac{\alpha + \beta + \gamma}{n} - 1, 1 < p, q, r < \infty, \frac{1}{r} \leq \frac{1}{p} + \frac{1}{q},$
- (2) $\alpha < \frac{n}{p'}, \beta < \frac{n}{q'}, \gamma < \frac{n}{r},$
- (3) $\alpha + \beta \geq 0, \beta + \gamma \geq 0, \gamma + \alpha \geq 0.$

Theorem 1.2. [6, Theorem 4.1]

$$L(p_0, q_0; \alpha) * L(p_1, q_1; \beta) \subset L(p, q; -\gamma)$$

provided

- (1) $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1} + \frac{\alpha + \beta + \gamma}{n} - 1, 1 < p_0, p_1, p < \infty, \frac{1}{p} \leq \frac{1}{p_0} + \frac{1}{p_1} \leq 1,$
- (2) $\alpha < \frac{n}{p_0'}, \beta < \frac{n}{p_1'}, \gamma < \frac{n}{p},$
- (3) $\alpha + \beta > 0, \beta + \gamma > 0, \gamma + \alpha > 0.$

Further weighted inequalities for convolutions can be found in [1, 7, 8, 10] (see also references therein). However, the fact that one can improve Theorems 1.1 and 1.2 when the functions involved are assumed to be radial was seemingly overlooked, and is the object of the present paper. Namely, we will prove:

Theorem 1.3.

$$L_{rad}(p; \alpha) * L_{rad}(q; \beta) \subset L_{rad}(r; -\gamma)$$

provided

- (1) $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} + \frac{\alpha + \beta + \gamma}{n} - 1, 1 < p, q, r < \infty, \frac{1}{r} \leq \frac{1}{p} + \frac{1}{q},$
- (2) $\alpha < \frac{n}{p'}, \beta < \frac{n}{q'}, \gamma < \frac{n}{r},$
- (3) $\alpha + \beta \geq (n-1)(1 - \frac{1}{p} - \frac{1}{q}), \beta + \gamma \geq (n-1)(\frac{1}{r} - \frac{1}{q}), \gamma + \alpha \geq (n-1)(\frac{1}{r} - \frac{1}{p})$
- (4) $\max\{\alpha, \beta, \gamma\} \geq 0.$

Theorem 1.4.

$$L_{rad}(p_0, q_0; \alpha) * L_{rad}(p_1, q_1; \beta) \subset L_{rad}(p, q; -\gamma)$$

provided

- (1) $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1} + \frac{\alpha+\beta+\gamma}{n} - 1$, $1 < p_0, p_1, p < \infty$, $0 \leq \frac{1}{q} \leq \frac{1}{q_0} + \frac{1}{q_1} \leq 1$,
- (2) $\alpha < \frac{n}{p_0}$, $\beta < \frac{n}{p_1}$, $\gamma < \frac{n}{p}$,
- (3) $\alpha + \beta > (n-1)(1 - \frac{1}{p_0} - \frac{1}{p_1})$, $\beta + \gamma > (n-1)(\frac{1}{p} - \frac{1}{p_1})$, $\gamma + \alpha > (n-1)(\frac{1}{p} - \frac{1}{p_0})$,
- (4) $\max\{\alpha, \beta, \gamma\} > 0$.

Remark 1.1. *Theorem 1.3 also holds for $r = 1$ and will be proved separately (see Theorem 2.2). Moreover, it can be seen from the proof of Theorem 1.3, that it also holds for $p = 1$ (or $q = 1$, but not both), provided the inequalities in $\alpha + \beta$, $\beta + \gamma$ and $\alpha + \gamma$ are strict and $\beta < (n-1)(\frac{1}{r} - \frac{1}{p})$ (respectively, $\alpha < (n-1)(\frac{1}{r} - \frac{1}{p})$).*

Notice that Theorem 1.3 is indeed an extension of Theorem 1.1 since in that case at most one among α, β, γ can be negative. Moreover, if, for instance, $1 - \frac{1}{p} - \frac{1}{q} > 0$ one has that, in both theorems,

$$\alpha + \beta = n \left(1 - \frac{1}{p} - \frac{1}{q} + \frac{1}{r} - \frac{\gamma}{n} \right) > n \left(1 - \frac{1}{p} - \frac{1}{q} \right) > (n-1) \left(1 - \frac{1}{p} - \frac{1}{q} \right)$$

so that our condition on $\alpha + \beta$ admits negative values and is only apparently more restrictive when the sum is non negative. Similar considerations apply also to $\alpha + \gamma$ and $\beta + \gamma$, and in the case of Theorem 1.4 compared to Theorem 1.2.

The restriction to radially symmetric functions is natural, for instance, for applications to partial differential equations in \mathbb{R}^n , but one could also be tempted to ask whether one can improve the conditions on $\alpha + \beta$, $\alpha + \gamma$ and $\beta + \gamma$ of Theorem 1.1 by restricting the convolution to functions invariant with respect to a different subgroup of the orthogonal group. However, the answer is negative, as we will show below (see Remark 2.1).

The rest of the paper is as follows. In Section 2 we prove some preliminary results; in Section 3 we prove Theorem 1.3 and in Section 4 we outline the proof of Theorem 1.4 and obtain, as a corollary, weighted estimates for fractional integrals of radial functions in Lorentz spaces.

2. PRELIMINARY RESULTS

First we show that the conditions of Theorem 1.1 on $\alpha + \beta$, $\beta + \gamma$ and $\alpha + \gamma$ cannot be improved for arbitrary functions, or indeed for any set of functions invariant with respect to a subgroup of the orthogonal group other than the radial functions. This can be done by applying the convolution inequality to the heat kernel, as was done in [2] to prove the necessity of the scaling and integrability conditions (see [2, Theorem 2.1]).

Remark 2.1. *If $L(p; \alpha) * L(q; \beta) \subset L(r; -\gamma)$, then $\alpha + \beta \geq 0$, $\alpha + \gamma \geq 0$ and $\beta + \gamma \geq 0$. The latter conditions are also necessary if the inequality is restricted to functions invariant with respect to any subgroup $G \subset O(n)$ such that there exist some y_0 , $|y_0| = 1$, fixed by the action of G , that is,*

$$g \cdot y_0 = y_0 \quad \forall g \in G.$$

Proof. One can verify easily that $L(p; \alpha) * L(q; \beta) \subset L(r; -\gamma)$ implies the scaling condition

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} + \frac{\alpha + \beta + \gamma}{n} - 1$$

(or see [2] for proof). Given this, let $W_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$ be the heat kernel in \mathbb{R}^n and let $W_{t,y}(x) = W_t(x - y)$ for $y \in \mathbb{R}^n$. Then, we have $W_{t,y} * W_{t,-y} = W_{2t}$, whence

$$\|W_{2t}\|_{r;-\gamma} \leq C \|W_{t,y}\|_{p;\alpha} \|W_{t,-y}\|_{q;\beta}.$$

Now, setting $z = \frac{x}{\sqrt{2t}}$, we have that

$$\|W_{2t}\|_{r;-\gamma} = (2t)^{(-n-\gamma+n/r)/2} \|W_1\|_{r;-\gamma} = t^{(-n-\gamma+n/r)/2} C(n, \gamma, r).$$

Similarly, setting $z = \frac{x-y}{\sqrt{t}}$,

$$\|W_{t,y}\|_{p;\alpha} = t^{(-n+\alpha+n/p)/2} \left(\int_{\mathbb{R}^n} |W_1(z)|^p \left| z + \frac{y}{\sqrt{t}} \right|^{\alpha p} dz \right)^{1/p},$$

and setting $z = \frac{x+y}{\sqrt{t}}$,

$$\|W_{t,-y}\|_{q;\beta} = t^{(-n+\beta+n/q)/2} \left(\int_{\mathbb{R}^n} |W_1(z)|^q \left| z - \frac{y}{\sqrt{t}} \right|^{\beta q} dz \right)^{1/q}.$$

Therefore, noting that the powers of t cancel out due to the scaling condition, one has that, for some $C > 0$ depending only on the parameters $p, q, r, \alpha, \beta, \gamma$ and the dimension n ,

$$C \leq \left(\int_{\mathbb{R}^n} |W_1(z)|^p \left| z + \frac{y}{\sqrt{t}} \right|^{\alpha p} dz \right)^{1/p} \left(\int_{\mathbb{R}^n} |W_1(z)|^q \left| z - \frac{y}{\sqrt{t}} \right|^{\beta q} dz \right)^{1/q}.$$

Choosing $y = \lambda \sqrt{t} y_0$ for some $y_0 \in \mathbb{R}^n$ fixed with $|y_0| = 1$, we get, for any $\lambda > 0$,

$$C \lambda^{-\alpha-\beta} \leq \left(\int_{\mathbb{R}^n} |W_1(z)|^p \left| \frac{z}{\lambda} + y_0 \right|^{\alpha p} dz \right)^{1/p} \left(\int_{\mathbb{R}^n} |W_1(z)|^q \left| \frac{z}{\lambda} - y_0 \right|^{\beta q} dz \right)^{1/q}$$

whence, letting $\lambda \rightarrow +\infty$, we deduce that $\alpha + \beta \geq 0$, since otherwise $\|W_1\|_p \|W_1\|_q \geq +\infty$, a contradiction.

In a similar way, using the relations

$$W_t * W_{t,y} = W_{2t,y}$$

and

$$W_{t,y} * W_t = W_{2t,y}$$

(or a duality argument) we can prove the necessity of the conditions $\beta + \gamma \geq 0$, $\alpha + \gamma \geq 0$.

In the case of functions invariant with respect to a subgroup $G \subset O(n)$ of the orthogonal group, such that there exist $y_0, |y_0| = 1$ fixed by the action of G , consider $y = \lambda \sqrt{t} y_0$. Then, as before, $W_{t,y}$, $W_{t,-y}$ and W_{2t} is a G -invariant function, so the same argument applies. \square

Now, note that a key point in the proof of Theorems 1.1 and 1.2 is the relationship between the convolution operator and the fractional integral, or Riesz potential, given by

$$(2.1) \quad (T_\gamma v)(x) = \int_{\mathbb{R}^n} \frac{v(y)}{|x-y|^\gamma} dy, \quad 0 < \gamma < n.$$

Indeed, the proof in [6] invokes known weighted estimates for this operator proved by E. Stein and G. Weiss [12]. We will follow that method of proof but use instead the following result proved by the authors and R.G. Durán in [4, Theorem 1.2],

that gives the weighted inequalities when T_γ is restricted to radial functions. The result for $p > 1$ was previously proved by different means in [11, Theorem 1.2]. An alternative proof for all $p \geq 1$ can also be found in [5, Theorem 5.1], where also sharpness of the result is proved.

Theorem 2.1. [4, Theorem 1.2] *Let $n \geq 1$, $0 < \gamma < n$, $1 < p < \infty$, $\alpha < \frac{n}{p'}$, $\beta < \frac{n}{q}$, $\alpha + \beta \geq (n-1)(\frac{1}{q} - \frac{1}{p})$, and $\frac{1}{q} = \frac{1}{p} + \frac{\gamma + \alpha + \beta}{n} - 1$. If $p \leq q < \infty$, then the inequality*

$$\|T_\gamma v\|_{q;-\beta} \leq C \|v\|_{p;\alpha}$$

holds for all radially symmetric $v \in L^p(\mathbb{R}^n, |x|^{p\alpha} dx)$, where C is independent of v . If $p = 1$, then the result holds provided $\alpha + \beta > (n-1)(\frac{1}{q} - 1)$.

Once we establish Theorem 1.4, we will also be able to extend this result to Lorentz spaces. However, for now we postpone a proof of this fact and prove that, as an almost immediate consequence of the previous theorem, one has the following special case of our convolution inequalities, that we will need later:

Theorem 2.2. *If, for $1 < p, q < \infty$, we have*

$$2 = \frac{1}{p} + \frac{1}{q} + \frac{\alpha + \beta + \gamma}{n} \quad , \quad \frac{1}{p} + \frac{1}{q} \geq 1 \quad ,$$

$$\alpha < \frac{n}{p'} \quad , \quad \beta < \frac{n}{q'} \quad , \quad 0 < \gamma < n \quad ,$$

and

$$(2.2) \quad \alpha + \beta \geq (n-1) \left(1 - \frac{1}{p} - \frac{1}{q} \right) ,$$

then

$$L_{rad}(p; \alpha) * L_{rad}(q; \beta) \subset L_{rad}(1; -\gamma).$$

The result also holds for $p = 1$ (or $q = 1$) provided that the inequality in (2.2) is strict.

Proof. It suffices to consider the case $f, g \geq 0$ and f radially symmetric. Then, by Tonelli's theorem,

$$\int_{\mathbb{R}^n} (f * g)(x) |x|^{-\gamma} dx = \int_{\mathbb{R}^n} (T_\gamma f)(x) g(-x) dx$$

whence, by Hölder's inequality and Theorem 2.1, given our conditions, we obtain

$$\|f * g\|_{1;-\gamma} \leq C \|T_\gamma f\|_{q';-\beta} \|g\|_{q;\beta} \leq C \|f\|_{p;\alpha} \|g\|_{q;\beta}$$

□

The above result coincides with Theorem 1.3 for $r = 1$. The result for other values of r will be obtained by duality and multilinear interpolation, we make explicit in which way in the next two results.

Lemma 2.1. *Suppose a and b are real numbers and $1 < r \leq \infty$, $1 \leq s < \infty$. Let f, g be nonnegative functions on \mathbb{R}^n , f radially symmetric, and define the linear operator T_f by*

$$(T_f g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y) dy.$$

Then,

$$T_f : L_{rad}(r; a) \rightarrow L_{rad}(s; b),$$

with operator norm C implies

$$T_f : L_{rad}(s'; -b) \rightarrow L_{rad}(r'; -a)$$

with the same norm.

Proof. It follows easily by using duality, we leave details to the reader. \square

Theorem 2.3 ([3]). *Suppose T is a multilinear operator satisfying*

$$T : L(p_i, w_i) \times L(p'_i, w'_i) \rightarrow L(p''_i, w''_i)$$

with norm $K_i, i = 0, 1$. Then,

$$T : L(p_t, w_t) \times L(p'_t, w'_t) \rightarrow L(p''_t, w''_t)$$

with norm at most $K_0^{1-t} K_1^t$, where $p_t, p'_t, p''_t, w_t, w'_t$ and w''_t are given by

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1} \quad \text{and} \quad w_t = w_0^{p_t(1-t)/p_0} w_1^{p_t t/p_1}.$$

Remark 2.2. When $w_i = |x|^{\alpha_i p_i}, w'_i = |x|^{\alpha'_i p'_i}, w''_i = |x|^{\alpha''_i p''_i}$, clearly one has $\alpha_t = (1-t)\alpha_0 + t\alpha_1$, $\alpha'_t = (1-t)\alpha'_0 + t\alpha'_1$ and $\alpha''_t = (1-t)\alpha''_0 + t\alpha''_1$.

Remark 2.3. Since we will actually use the above theorem to interpolate between the subspaces of radial functions of the corresponding spaces, a comment is in order. In general, one cannot freely interpolate between subspaces and guarantee that the intermediate space is the expected subspace. However, the subspaces $L_{rad}(p, \alpha)$ in \mathbb{R}^n are isomorphic to spaces $L(p, \alpha + \frac{n-1}{p})$ in $(0, \infty)$, so one may interpolate in the latter setting and use the fact that the interpolation commutes with the standard isomorphism.

3. CONVOLUTION IN WEIGHTED LEBESGUE SPACES

In this section we prove Theorem 1.3. First we claim the following:

Remark 3.1. *Given Theorem 1.1, it suffices to prove that Theorem 1.3 holds in the following cases:*

- (1) $\alpha + \beta \leq 0, \gamma > 0$
- (2) $\beta + \gamma \leq 0, \alpha > 0$
- (3) $\alpha + \gamma \leq 0, \beta > 0$

Proof. Consider first $\gamma > 0$. Then, if $\alpha + \beta \leq 0$ the theorem will follow from case (1). If $\alpha + \beta > 0$ there are two possibilities: either $\alpha, \beta > 0$ or one of them, say β , is negative. In the first case, the result follows from Theorem 1.1. In the second case, $\alpha > -\beta > 0$ and we have again two possibilities. If $\gamma + \beta > 0$ the result follows from Theorem 1.1 since, under our assumptions, we also have $\alpha + \beta, \alpha + \gamma > 0$. If $\gamma + \beta < 0$, that is, $\alpha > -\beta > \gamma \geq 0$ then we have that $\beta + \gamma < 0$ and $\alpha > 0$, so the theorem will follow from case (2).

Now assume that $\gamma \leq 0$. In this case, if $\alpha, \beta > 0$, three things can happen. If $\alpha + \gamma \geq 0$ and $\beta + \gamma \geq 0$, the result follows from Theorem 1.1. Otherwise, one has that either $\alpha + \gamma < 0$ or $\beta + \gamma < 0$, in which cases the result follows from cases (3) or (2), respectively.

There remain the cases $\gamma \leq 0, \alpha > 0, \beta \leq 0$, and $\gamma \leq 0, \alpha \leq 0, \beta > 0$ which are contained in cases (2) and (3), respectively. \square

Proof. (Theorem 1.3). We begin by considering the case $\alpha + \beta \leq 0, \gamma > 0$.

Now, $\alpha + \beta \leq 0$ clearly implies $\frac{1}{p} + \frac{1}{q} \geq 1$, so this case is analogous to the first case of the proof of Theorem 1.1 given in [6], and the result follows by interpolation between Hölder's inequality and Theorem 2.2, that is, using Theorem 2.3 with the endpoints

$$r_0 = \infty, \quad \frac{1}{p_0} + \frac{1}{q_0} = 1, \quad \alpha_0 = \beta_0 = \gamma_0 = 0$$

and

$$\begin{aligned} r_1 = 1, \quad 2 = \frac{1}{p_1} + \frac{1}{q_1} + \frac{\alpha_1 + \beta_1 + \gamma_1}{n}, \quad \frac{1}{p_1} + \frac{1}{q_1} \geq 1 \\ \alpha_1 < \frac{n}{p'_1}, \quad \beta_1 < \frac{n}{q'_1}, \quad 0 < \gamma_1 < n \\ \alpha_1 + \beta_1 \geq (n-1) \left(1 - \frac{1}{p_1} - \frac{1}{q_1} \right). \end{aligned}$$

Hence, $t = \frac{1}{r}$ and $\alpha_1 = r\alpha, \beta_1 = r\beta, \gamma_1 = r\gamma$.

Clearly $0 < \gamma_1 < n$. The remaining conditions depend on the choice of p_0 . We begin by considering $\beta_1 < \frac{n}{q'_1}$, which is equivalent to

$$\frac{1}{p_0} < \frac{1 - \frac{1}{q} - \frac{\beta}{n}}{1 - \frac{1}{r}}.$$

If the right hand side is greater than 1, that is $\frac{1}{r} - \frac{1}{q} > \frac{\beta}{n}$, we choose $p_0 = 1$. Then, $q_0 = \infty, \frac{1}{p_1} = r(\frac{1}{p} + \frac{1}{r} - 1)$ and $\frac{1}{q_1} = \frac{r}{q}$. Given this, one can check that condition 2 = $\frac{1}{p_1} + \frac{1}{q_1} + \frac{\alpha_1 + \beta_1 + \gamma_1}{n}$ follows from $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} + \frac{\alpha + \beta + \gamma}{n} - 1$, that condition $\alpha_1 < \frac{n}{p'_1}$ follows from $\alpha < \frac{n}{p'}$, that condition $\frac{1}{p_1} + \frac{1}{q_1} \geq 1$ follows from $\frac{1}{p} + \frac{1}{q} \geq 1$, and that condition $\alpha_1 + \beta_1 \geq (n-1)(1 - \frac{1}{p_1} - \frac{1}{q_1})$ follows from $\alpha + \beta \geq (n-1)(1 - \frac{1}{p} - \frac{1}{q})$.

If $\frac{1}{r} - \frac{1}{q} \leq \frac{\beta}{n}$, we choose $\frac{1}{p_0} = (1 - \varepsilon)(1 - \frac{1}{q} - \frac{\beta}{n}) / (1 - \frac{1}{r})$ for small positive ε that we will choose later. Then, $\frac{1}{q_0} = [\frac{1}{q} + \frac{\beta}{n} - \frac{1}{r} + \varepsilon(1 - \frac{1}{q} - \frac{\beta}{n})] / (1 - \frac{1}{r})$, $\frac{1}{p_1} = r[\frac{1}{p} - (1 - \varepsilon)(1 + \frac{1}{q} + \frac{\beta}{n})]$ and $\frac{1}{q_1} = r[\frac{1}{r} - \frac{\beta}{n} - \varepsilon(1 - \frac{1}{q} - \frac{\beta}{n})]$. Therefore, condition $\alpha_1 < \frac{n}{p'_1}$ follows from the scaling condition and the fact that $\gamma > 0$, provided we choose $\varepsilon < \gamma / (\frac{n}{q} - \beta)$, condition $\frac{1}{p_1} + \frac{1}{q_1} \geq 1$ follows from $\frac{1}{p} + \frac{1}{q} \geq 1$, condition 2 = $\frac{1}{p_1} + \frac{1}{q_1} + \frac{\alpha_1 + \beta_1 + \gamma_1}{n}$ follows from $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} + \frac{\alpha + \beta + \gamma}{n} - 1$ and condition $\alpha_1 + \beta_1 \geq (n-1)(1 - \frac{1}{p_1} - \frac{1}{q_1})$ follows from $\alpha + \beta \geq (n-1)(1 - \frac{1}{p} - \frac{1}{q})$.

This completes the case $\alpha + \beta \leq 0, \gamma > 0$ and, by Lemma 2.1, one has then the result for $\gamma + \beta \leq 0, \alpha > 0$ and $\alpha + \gamma \leq 0, \beta > 0$, which in view of Remark 3.1 complete the proof. \square

Remark 3.2. If $p = 1$, as mentioned in Remark 1.1, one can choose $p_0 = p_1 = 1$ above, since obviously $\frac{1}{p} + \frac{1}{q} \geq 1$. However, in this case one needs $\beta < (n-1)(\frac{1}{r} - \frac{1}{q})$ to follow through the proof. By symmetry, if $q = 1$, one needs $\alpha < (n-1)(\frac{1}{r} - \frac{1}{p})$.

4. CONVOLUTION IN WEIGHTED LORENTZ SPACES

Proof. (Theorem 1.4). The proof can be carried out exactly as the proof of Theorem 1.2 given in [6], once we establish that, under the assumptions of the theorem, the

restricted weak type inequality

$$\begin{aligned} \int_H (\chi_F * \chi_G)(x) |x|^{-\gamma p} dx \\ \leq C \left(\int_F |x|^{\alpha p_0} dx \right)^{\frac{1}{p_0}} \left(\int_G |x|^{\beta p_1} dx \right)^{\frac{1}{p_1}} \left(\int_H |x|^{\gamma p} dx \right)^{\frac{1}{p'}} \end{aligned}$$

holds for $F, G, H \subset \mathbb{R}^n$ of finite measure, such that χ_F and χ_G are radial. Indeed, if $\frac{1}{p} \leq \frac{1}{p_0} + \frac{1}{p_1}$, by Hölder's inequality

$$\int_H (\chi_F * \chi_G)(x) |x|^{-\gamma p} dx \leq C \|\chi_F * \chi_G\|_{p; -\gamma} \left(\int_H |x|^{-\gamma p} dx \right)^{\frac{1}{p'}}$$

and the result follows by Theorem 1.3. If instead one has $\frac{1}{p} > \frac{1}{p_0} + \frac{1}{p_1}$, the result holds even for non necessarily radial functions and is contained in [6, Proposition 4.2].

Since the rest of the proof is as in [6], noticing again that one may interpolate between radial subspaces for similar reasons as those of Remark 2.3, we leave the details to the reader. \square

As an application of Theorem 1.4 one has the following result for Riesz potentials (defined by (2.1)) of radial functions in Lorentz spaces with power weights, that extends the result obtained in [6, Theorem 4.5] for non necessarily radial functions in a similar way.

Theorem 4.1. *Let $0 < \lambda < n$, $1 < p_0 < \infty$, $\alpha < \frac{n}{p_0}$, $\gamma < \frac{n}{p}$, $\alpha + \gamma > (n-1)(\frac{1}{p} - \frac{1}{p_0})$ and $\frac{1}{p} = \frac{1}{p_0} + \frac{\alpha + \lambda + \gamma}{n} - 1$. Then,*

$$T_\lambda : L_{rad}(p_0, q_0; \alpha) \rightarrow L_{rad}(p, q; -\gamma)$$

for $q \geq q_0$.

Proof. Assume v radial. Then, by Theorem 1.4,

$$\|T_\lambda v\|_{p, q; -\gamma} \leq C \| |x|^{-\lambda} \|_{p_1, q_1; \beta} \|v\|_{p_0, q_0; \alpha}$$

provided we also have $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1} + \frac{\alpha + \beta + \gamma}{n} - 1$, $\beta < \frac{n}{p_1}$, $\alpha + \beta > (n-1)(1 - \frac{1}{p_0} - \frac{1}{p_1})$, $\beta + \gamma > (n-1)(\frac{1}{p} - \frac{1}{p_1})$ and $\max\{\alpha, \beta, \gamma\} > 0$.

We claim that it suffices to take $q_1 = \infty$, $\frac{1}{p_1} < \min\{\frac{1}{p_0}, \frac{1}{p}\}$ and let $\frac{\beta}{n} = \frac{\lambda}{n} - \frac{1}{p_1}$. Indeed, with this choice of parameters, $\| |x|^{-\lambda} \|_{p_1, q_1; \beta} < \infty$ and $\frac{1}{p} = \frac{1}{p_0} + \frac{\alpha + \lambda + \gamma}{n} - 1$. Moreover, by the conditions on α and γ and the choice of p_1 ,

$$\alpha + \beta = n \left(\frac{1}{p} - \frac{1}{p_0} - \frac{1}{p_1} - \frac{\gamma}{n} + 1 \right) > (n-1) \left(1 - \frac{1}{p_0} - \frac{1}{p_1} \right)$$

and

$$\beta + \gamma = n \left(\frac{1}{p} - \frac{1}{p_0} - \frac{1}{p_1} - \frac{\alpha}{n} + 1 \right) > (n-1) \left(\frac{1}{p} - \frac{1}{p_1} \right).$$

Finally, since by the condition on p_1

$$\frac{\alpha + \beta + \gamma}{n} = \frac{1}{p} + \frac{1}{p_0} - \frac{1}{p_1} > 0,$$

one clearly has $\max\{\alpha, \beta, \gamma\} > 0$, as required. \square

Acknowledgements. The authors thank Carlos D’Andrea for reference [2].

REFERENCES

- [1] Biswas, A.; Swanson, D. *Navier-Stokes equations and weighted convolution inequalities in groups*. Comm. Partial Differential Equations 35 (2010), no. 4, 559–589.
- [2] Bui, H-Q. *Weighted Young’s inequality and convolution theorems on weighted Besov spaces*. Math. Nachr. 170 (1994), 25–37.
- [3] Calderón, A.-P. *Intermediate spaces and interpolation, the complex method*. Studia Math. 24 (1964) 113–190.
- [4] De Nápoli, P.L. ; Drelichman, I.; Durán, R.G., *On weighted inequalities for fractional integrals of radial functions*. Illinois J. Math., to appear.
- [5] J. Duoandikoetxea. *Fractional integrals on radial functions with applications to weighted inequalities*, Ann. Mat. Pura Appl. (4), DOI 10.1007/s10231-011-0237-7, (to appear in print).
- [6] Kerman, R. A. *Convolution theorems with weights*. Trans. Amer. Math. Soc. 280 (1983), no. 1, 207–219.
- [7] Kerman, R.; Sawyer, E. *Convolution algebras with weighted rearrangement-invariant norm*. Studia Math. 108 (1994), no. 2, 103–126.
- [8] Nursultanov, E.; Tikhonov, S. *Convolution inequalities in Lorentz spaces*. J. Fourier Anal. Appl. 17 (2011), no. 3, 486–505.
- [9] O’Neil, R. *Convolution operators and $L(p, q)$ spaces*. Duke Math. J. 30 (1963), 129–142.
- [10] Rakotondratsimba, Y. *Weighted Young inequalities for convolutions*. Southeast Asian Bull. Math. 26 (2002), no. 1, 77–99.
- [11] Rubin, B.S. *One-dimensional representation, inversion and certain properties of Riesz potentials of radial functions* (Russian), Mat. Zametki 34 (1983), no. 4, 521–533. English translation: Math. Notes 34 (1983), no. 3–4, 751–757.
- [12] Stein, E. M.; Weiss, G. *Fractional integrals on n -dimensional Euclidean space*. J. Math. Mech. 7 (1958), 503–514.

IMAS (UBA-CONICET) AND DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS EXACTAS Y NATURALES, UNIVERSIDAD DE BUENOS AIRES, CIUDAD UNIVERSITARIA, 1428 BUENOS AIRES, ARGENTINA

E-mail address: pdenapo@dm.uba.ar

IMAS (UBA-CONICET) AND DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS EXACTAS Y NATURALES, UNIVERSIDAD DE BUENOS AIRES, CIUDAD UNIVERSITARIA, 1428 BUENOS AIRES, ARGENTINA

E-mail address: irene@drelichman.com